ON THE REPRESENTATIONS OF NUMBERS AS SUMS OF 3, 5, 7, 9, 11 AND 13 SQUARES.*

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I. Preliminary Considerations.

1. Of the nine sections into which this paper is divided, I is preliminary; II contains eleven general formulas of a simple nature; III–VIII apply the first seven formulas of II to the determination in finite form of the number of representations of an integer as a sum of 3, 5, 7, 9, 11 and 13 squares respectively, attention being paid to the numbers of odd squares in the representations; and IX is devoted to recurrences, consequences of the last four formulas of II, for facilitating the numerical computations implied in III–VIII. A complete system of results of a well-defined kind, described in §§ 4–6, is obtained in III–VIII; but in the sense of determining finitely the total number of representations for any linear form of the integer to be represented, the enumerations for 11 and 13 squares are only partial. If complete finite systems are desired in these and further odd cases, they may be found by applying II to the theorems of H. J. S. Smith, Glaisher, and others, concerning an even number of squares. The resulting formulas, however, are simple neither in the common meaning nor in the technical sense defined in § 4; and as they belong to a wholly different order of ideas, they are not included here.

From one point of view the formulas, particularly the recurrences, for 5, 7, 9, 11 and 13 squares may be looked upon as analogues of the class number relations of Kronecker,† which, as has long been known, are intimately related to the decompositions of integers into sums of three squares. In fact, if the incomplete primitives of § 8, which have been expressly avoided here, are admitted, all of Kronecker's results and more of the same kind due to later writers reappear; and it appears that the 5, 7, 9, 11, 13 square theorems are of the same general nature. To find closer analogues it is only necessary to replace the functions denoting the numbers of representations as sums of 5, 7, $\cdots$ squares by their weight equivalents as determined by H. J. S. Smith in his memoir "On the Orders and Genera of Quadratic Forms Containing More than Three Indeterminates." The re-

* Read before the San Francisco Section of the American Mathematical Society, April 5, 1919.

† The development of these and other relations from the present point of view will be published elsewhere.

sults are most numerous for 9 squares. The recurrences evaporate with the case of 11 squares. Taken together, all form a short natural chapter in arithmetic.

In counting representations we include, as customary, both the order of the squares and the signs of their square roots. Thus the single decomposition of 54 into a sum of 9 squares,

$$54 = 1^2 + 2^2 + 2^2 + 3^2 + 3^2 + 3^2 + 3^2 + 3^2 + 0^2,$$

contributes $2^8 \cdot 9!/2!5! = 387072$ representations.

As constant use is made of Glaisher's work for 2, 4, $\cdots$, 12 squares, we may cover all references to it by this citation: *Quarterly Journal,* 38 (1906–7), pp. 1–68. A convenient précis of the square-theorem results of his paper is given in the *Proceedings of the London Mathematical Society,* (2) 5 (1907), pp. 479–490.

2. All letters $m, \mu, n, a, b, c, r, s,$ denote positive integers, different from zero unless explicitly noted to the contrary; m, μ are always odd, and the rest, unless further specified, arbitrary; k is an integer $\geqq 0$. We define $f(n)$ to be primitive if its values may be calculated in finite form from the real divisors of n alone, and consider $f(x) = 0$ (or, if preferred, non-existent), when $x \lesseqgtr 0$, or when x is not an integer. With these conventions, the respective sums

$$f(n-4) + f(n-16) + f(n-36) + f(n-64) + \cdots,$$

$$f(n-1) + f(n-9) + f(n-25) + f(n-49) + \cdots,$$

consist of only a finite number of terms, and may be written $\Sigma f(n - 4a^2)$, $\Sigma f(n - \mu^2)$, the Σ extending only to those values of a, μ that render $n - 4a^2$, $n - \mu^2 > 0$. Similarly for sums of functions of the triangular numbers, 1, 3, 6, 10, $\cdots$, such as

$$f(n-4) + f(n-12) + f(n-24) + f(n-40) + \cdots,$$

which may be denoted by $\Sigma f(n - 4t)$, t representing, as always throughout the paper, a triangular number.

By repeated application of the obvious identity

$$\Sigma f(n - a^2) = \Sigma[f(n - \mu^2) + f(n - 4a^2)],$$

we get a transformation which frequently is useful:

$$\Sigma f(n - a^2) = \Sigma[f(n - 4^s a^2) + \sum_{r=0}^{s-1} f(n - 4^r \mu^2)].$$

3. We shall require the primitives: $\zeta_g(n)$, = the sum of the gth powers of all the divisors of n; $\zeta_g'(n)$, $\zeta_g''(n)$ the like for the odd, even divisors

respectively; $\xi_g(n)$, = the excess of the sum of the gth powers of all those divisors of n that are $\equiv 1 \bmod 4$ over the like sum for the divisors $\equiv -1 \bmod 4$; $\xi'_g(n)$, = the excess of sum of the gth powers of all those divisors of n whose conjugates are $\equiv 1 \bmod 4$ over the like sum for the divisors whose conjugates are $\equiv -1 \bmod 4$; $\epsilon(n) = \epsilon(a^2 n)$, = 1 or 0 according as n is or is not a square. From these we construct further primitives as required. When $g = 0$, $\zeta_g, \cdots, \xi'_g$ are written $\zeta, \cdots, \xi'$ respectively, and denote the numbers of divisors in the respective classes defined by the corresponding functions. Our notation follows Liouville's, to conform with other papers on similar topics. Glaisher's notation is given loc. cit., pp. 3–6.

4. If $f(n)$ is primitive, then a sum of the form $\Sigma f[(pn - qa^2)/g]$, in which p, q, g are numerical constants, n is an integer, and the summation is with respect to the variable integer a, is defined to be simple. Thus, $\Sigma f(n - a^2)$ is a simple sum, and its value being determined when n is given, we shall call $\Sigma f(n - a^2)$ a simple function of n; and, by a legitimate extension, say that any linear function of a finite number of simple functions of n is a simple function of n. The simple functions most frequently occurring hereafter are of the forms $\Sigma f[(m - \mu^2)/g]$, where $g = 1, 2, 4$ or 8; $\Sigma f(n - a^2)$; $\Sigma f(m - 4a^2)$.

5. Let $N_r(n)$, $N_r(n, g)$ denote respectively the total number of representations of n as a sum of r squares, and the total number of representations of n as a sum of r squares precisely g of which are odd. Then clearly $N_r(n)$ is expressible in the form

$$\sum_{g=0}^{r} c_g N_r(n, g),$$

wherein $c_g = 1$ or 0. When the linear form, modulo s, of n is assigned, the constants c_g are known. We shall use $s = 8$; hence, observing that $m^2 \equiv 1 \bmod 8$, $(2a)^2 \equiv 0$ or $4 \bmod 8$, we have for the determination of g,

$$g + \alpha(r - g) \equiv n \bmod 8, \qquad (\alpha = 0, 4); \qquad 0 \leqq g \leqq r.$$

When r is specified we take in this way a census of the possible linear forms modulo 8. For a particular r the census is most readily found by inspection. Thus, for $r = 11$, (cf. the m, n, k-notation in § 2):

$$\begin{aligned}
n &= 4k &&: N_{11}(n) = N_{11}(n, 0) + N_{11}(n, 4) + N_{11}(n, 8),\\
m &= 4k + 1 &&: N_{11}(m) = N_{11}(m, 1) + N_{11}(m, 5) + N_{11}(m, 9),\\
n &= 2m &&: N_{11}(n) = N_{11}(n, 2) + N_{11}(n, 6) + N_{11}(n, 10),\\
m &= 8k + 3 &&: N_{11}(m) = N_{11}(m, 3) + N_{11}(m, 7) + N_{11}(m, 11),\\
m &= 8k + 7 &&: N_{11}(m) = N_{11}(m, 3) + N_{11}(m, 7).
\end{aligned}$$

Hence, for example, knowing $N_{11}(m, 1)$, $N_{11}(m, 5)$, $N_{11}(m, 9)$ when $m \equiv 1$ or 5 mod 8, we can write down the value of $N_{11}(m)$. The census also shows what values of $N_r(n, g)$ vanish; thus, $N_{11}(2m, 8) = 0$, $N_{11}(2m, 4) = 0$.

6. We shall seek to determine all n, g such that

$$N_r(n), \qquad N_r(n, g), \qquad (r = 3, 5, 7, 9, 11, 13)$$

are simple functions of n, and to exhibit a set of simple functions giving the values of $N_r(n)$, $N_r(n, g)$ in these cases. It will appear that there is not a unique set, for the functions in any set may be transformed in many ways by means of the elementary identities existing between the primitives of § 3. On equating different expressions of the same $N_r(n)$ or $N_r(n, g)$ we get, in several instances, rapid recurrences for the successive calculation of the primitives involved; and in all cases the formulas obtained are well adapted to numerical computation. We may state here, reserving full discussion for another occasion, that when linear forms only are considered, $N_r(n)$ is simple for no n when $r = 15, 17, 19, 21, 23, 25$, and probably for no odd $r > 25$; the same applies to $N_r(n, g)$; so that the formulas of this paper form a natural class.

7. To compare the expressions through simple functions with the results given by the classical theory, let us consider, from the standpoint of their adaptability to numerical computation, the three following, of which (A) is due to Eisenstein,* (B) to Stieltjes,† and (C) is found in section IV.

$$(A) \qquad \lambda = 8k + 5: \qquad N_5(\lambda) \quad = -112 \sum_{s=1}^{[\lambda/2]} (s|\lambda)s;$$

$$(B) \qquad m = 8k + 5: \qquad N_5(m, 5) = 32\Sigma\zeta_1\left(\frac{m - \mu^2}{4}\right);$$

$$(C) \qquad m = 8k + 5: \qquad N_5(m) \quad = 112\Sigma\zeta_1\left(\frac{m - \mu^2}{4}\right).$$

In (A), λ is divisible by no square > 1; $[x]$ is the greatest integer in x, and $(s|\lambda)$ is the Legendre-Jacobi symbol, $(s|\lambda) = 0$ when s, λ are not relatively prime. In (B), (C), m is unrestricted, and the summation refers, by the conventions of § 2, to all odd μ such that $0 < \mu \leqq [\sqrt{m}]$. Let $m = 133$; the computation by (A) requires as a first step the calculation of the quadratic characters with respect to 133 of the 55 numbers 1, 2, 3, $\cdots$, 66 prime to 133 and < 67. For a large λ the s prime to λ would have to be determined in practically the same way as the $(s|\lambda)$, viz., by Eisenstein's or one of the equivalent algorithms for $(s|\lambda)$, which amounts to the

* Eisenstein, Crelle, 35 (1847), p. 368.

† Stieltjes, *Comptes Rendus*, 97 (1883), p. 981.

conversion of the s/λ into continued fractions, so that for λ not factorable by inspection there would be in all $[\lambda/2]$ such calculations. In that all of these may be performed non-tentatively, formulas such as (A) are superior to (B) or (C). But with the aid of a factor table, it would seem that even for λ fairly large, say $\lambda = 10005$, the 50 resolutions into prime factors required by (C) could be performed more expeditiously than the (approximately) 2500 conversions necessary in (A). Again, if it were required to construct tables, we should have the advantage of recurrences (found in section IX), such as

$$m = 8k + 5: \qquad N_5(m) = -14\xi_2\left(\frac{m+1}{2}\right) - \Sigma N_5(m - 8t),$$

and several similar relations between the primitives ξ_2 whereby their computation may be greatly abridged. For $m = 133$, we find from (C),

$$\begin{aligned} N_5(133) &= 112[\zeta_1(33) + \zeta_1(31) + \zeta_1(27) + \zeta_1(21) + \zeta_1(13) + \zeta_1(3)] \\ &= 112[48 + 32 + 40 + 32 + 14 + 4] = 19040. \end{aligned}$$

8. Functions that may be calculated from the divisors of n subject to one or more conditions were called by Hermite incomplete. Formulas analogous to those of this paper, but involving incomplete functions, may be found on starting from Liouville's '*formules gênérales,*' or from the elliptic theta equivalents of these was was done by Hermite* for $N_5(n)$ and $N_5(m, 5)$. He found

$$m = 8k + 5: \qquad N_5(m, 5) = G_1(n) + 2\Sigma G_1(m - 4a^2),$$

where $G_1(m)$ is defined by $G_1(m) = 4\Sigma(3d + d')$, the Σ extending to all positive integral solutions of $dd' = m$, $d' > 3d$. Incomplete functions have purposely been avoided in the sequel, as they appear to be less well adapted to computation than the primitives. In regard to Liouville's general formulas for which he did not publish proofs, and which, as indicated may be made to yield the results for 3, 5, $\cdots$, 13 squares, it was remarked by Hurwitz† that Stieltje's results possibly follow from some of them. It might seem, then, that all the simple functions should be derived directly from the same source, without the reference to elliptic functions implied by the use of Glaisher's results. But the reduction is only apparent, the origin of Liouville's formulas appearing most naturally in precisely those elliptic function identities that give the square theorems at once.

* Hermite, Oeuvres, IV, p. 238.

† Hurwitz, *Comptes Rendus*, 98 (1884), pp. 504–507. The two theorems which Liouville derives from his general formulas are insufficient for the proof of Stieltjes' results; cf. footnote to Section IV. Liouville did not indicate the connection between his theorems and representations as sums of 5 squares, nor did he carry out his intention of returning to the subject in a separate article.

II. GENERAL FORMULAS.

9. The notation has been explained in §§ 2, 5, and will be used as there given without further references. Summations being with respect to $\mu = 1, 3, 5, \cdots, \mu_1 = \pm 1, \pm 3, \pm 5, \cdots$, and $a = 1, 2, 3, \cdots, a_1 = 0, \pm 1, \pm 2, \pm 3, \cdots$, we have, in the usual notation of the elliptic theta constants,

$$\vartheta_2(q^4) = \Sigma q^{\mu_1^2} = 2\Sigma q^{\mu^2}, \tag{1}$$

$$\vartheta_3(q) = \Sigma q^{a_1^2} = 1 + 2\Sigma q^{a^2}. \tag{2}$$

Let $N'_r(n, s)$ denote the total number of representations of n as a sum of r squares, precisely s of which are odd and occupy the first s places in the representations; then, obviously

$$r!\, N'_r(n, s) = s!\,(r - s)!\, N_r(n, s), \tag{3}$$

$$\vartheta_2^s(q^4)\vartheta_3^{r-s}(q^4) = \sum_{n=1}^{\infty} q^n N'_r(n, s). \tag{4}$$

Consider the following identities, where $r > 1$:

$$\vartheta_2^{s+1}(q^4)\vartheta_3^{r-s-1}(q^4) = \vartheta_2(q^4) \times \vartheta_2^s(q^4)\vartheta_3^{r-s-1}(q^4), \tag{5}$$

$$\vartheta_2^s(q^4)\vartheta_3^{r-s}(q^4) = \vartheta_3(q^4) \times \vartheta_2^s(q^4)\vartheta_3^{r-s-1}(q^4), \tag{6}$$

$$\vartheta_2(q^4)\vartheta_3^{r-1}(q^4) = \vartheta_2(q^4) \times \vartheta_3^{r-1}(q^4), \tag{7}$$

$$\vartheta_3^r(q) = \vartheta_3(q) \times \vartheta_3^{r-1}(q). \tag{8}$$

By (4) the coefficient of q^n in the left member of (5) is $N'_r(n, s + 1)$. On using the second form of (1) for $\vartheta_2(q^4)$ on the right of (5), applying (4) to the second factor, and multiplying the resulting series (which are absolutely convergent), we find $2\Sigma N'_{r-1}(n - \mu^2, s)$ as the coefficient of q^n. Treating (6), (7), (8) similarly, equating coefficients of like powers of q, and using (3) to replace N' by its equivalent N, we find in this way from (5)–(8) the four fundamental identities, $r > 1$:

$$\text{(I)} \qquad (s + 1)N_r(n, s + 1) = 2r\Sigma N_{r-1}(n - \mu^2, s),$$

$$\text{(II)} \qquad (r - s)N_r(n, s) = r[N_{r-1}(n, s) + 2\Sigma N_{r-1}(n - 4a^2, s)],$$

$$\text{(III)} \quad m = 4k + 1:\ N_r(m, 1) = 2r\left[\epsilon(m) + \Sigma N_{r-1}\left(\frac{m - \mu^2}{4}\right)\right],$$

$$\text{(IV)} \qquad N_r(n) = 2\epsilon(n) + N_{r-1}(n) + 2\Sigma N_{r-1}(n - a^2).$$

Again, from the definitions, $\vartheta_3^r(q^4) = \Sigma q^{4n}N_r(4n, 0)$; whence, on changing q into $\sqrt[4]{q}$,

$$\vartheta_3^r(q) = \Sigma q^n N_r(4n, 0), = \Sigma q^n N_r(n),$$

the last by (2) and the definition of $N_r(n)$ in § 5. Hence

$$\text{(V)} \qquad N_r(4n, 0) = N_r(n).$$

10. It follows from (I), since $\mu^2 \equiv 1 \bmod 8$, that if $N_{r-1}(n, s)$ is primitive for $n \equiv g \bmod 8$, then $N_r(n, s+1)$ is simple when $n \equiv g+1 \bmod 8$. Similarly from (II), if $N_{r-1}(n, s)$ is primitive for $n \equiv g \bmod 4$, then $N_r(n, s)$ is simple for $n \equiv g \bmod 4$; and from (III), for $m \equiv 1 \bmod 4$, $N_r(m, 1)$ is simple if $N_{r-1}(n)$ is primitive, since $(m - \mu^2)/4$ may take any odd or even value, viz., it may take the values n; and the case (IV) shows that $N_r(n)$ is simple if $N_{r-1}(n)$ is primitive. Using (V) we get at once from (III), (IV) the corresponding forms for $N_r(m, 1)$ in terms of $N_{r-1}(m - \mu^2, 0)$, and of $N_r(4n, 0)$. Hence in sifting out the simples for a given r, we examine for what linear forms of n modulo 4 or 8, $N_{r-1}(n)$ is primitive, and apply the appropriate formulas of (I)–(V).

11. From (I), (II) we get, the c's being arbitrary constants:

$$\text{(VI)} \qquad \sum_i c_i(1 + s_i)N_r(n, 1 + s_i) = 2r\sum_i c_i[\Sigma N_{r-1}(n - \mu^2, s_i)],$$

$$\text{(VII)} \qquad \sum_i c_i(r - s_i)N_r(n, s_i) = r\sum_i c_i[N_{r-1}(n, s_i) + 2\Sigma N_{r-1}(n - 4a^2, s_i)].$$

It is a remarkable fact, first stated by Liouville,* that for r odd and $n \equiv 0$ or $2 \bmod 4$, there always exist integers c_i, s_i depending upon r but not upon n such that $\Sigma_i c_i N_{r-1}(n, s_i)$ is primitive. Moreover, the s_i are in arithmetical progression. Suitably choosing the c_i, s_i we may therefore find in certain cases linear functions, viz., the left members of (VI), (VII) that are simple functions of n, although in general the individual $N_r(n, 1 + s_i)$, $N_r(n, s_i)$ in the linear functions are neither primitive nor simple. To avoid reproducing the proofs of Liouville's theorems we shall write down the few necessary cases of (VI), (VII) directly from Glaisher's lists, whence they may be found by inspection.

12. The formulas (I)–(IV) may be reversed. In (I) change r into $r+1$, n into $n+1$, and solve for $N_r(n, s)$; in (II) replace r by $r+1$, and solve for $N_r(n, s)$; in (III) put $r+1$ for r, $4n+1$ for m, and solve for $N_r(n)$; and in (IV) change r into $r+1$ and solve for $N_r(n)$; then

$$\text{(I}'\text{)} \qquad 2(r+1)N_r(n, s) = (s+1)N_{r+1}(n+1, s+1) - 2(r+1)\Sigma N_r(n - 8t, s),$$

$$\text{(II}'\text{)} \qquad (r+1)N_r(n, s) = (r+1-s)N_{r+1}(n, s) - 2(r+1)\Sigma N_r(n - 4a^2, s),$$

$$\text{(III}'\text{)} \qquad 2(r+1)N_r(n) = N_{r+1}(4n+1, 1) - 2(r+1)[\epsilon(4n+1) + \Sigma N_r(n - 2t)],$$

$$\text{(IV}'\text{)} \qquad N_r(n) = N_{r+1}(n) - 2\epsilon(n) - 2\Sigma N_r(n - a^2).$$

* *Journal des Math.*, (2) 6 (1861), 2 papers, pp. 233–, 369–. The proofs were not given, nor a method for determining the c_i when r is given. Both were published in *Bull. Amer. Math. Soc.*, Oct., 1919.

Of these, (IV′) shows that if $N_{r+1}(n)$ is primitive, then the values of $N_r(n)$ may be calculated by recurrence from $N_r(n-1)$, $N_r(n-4)$, $N_r(n-9)$, $\cdots$. Similarly for (I′)–(III′); remembering that t always denotes a triangular number. In each case the value of a primitive for one value of the variable has to be calculated in addition to the N_r functions; thus in (IV′) the assumed primitive is $N_{r+1}(n)$.

III. THREE SQUARES.

13. We first take the census (§ 5) for 3 squares:

$$N_3(4k) = N_3(4k, 0); \quad N_3(4k+1) = N_3(4k+1, 1);$$

$$N_3(4k+2) = N_3(4k+2, 2); \; N_3(8k+3) = N_3(8k+3, 3); \; N_3(8k+7) = 0;$$

next listing the known theorems for 2 squares:

$$(1) \quad N_2(n) = 4\xi(n); \qquad (2) \quad N_2(m) = N_2(m, 1) = 4\xi(m);$$

$$(3) \quad N_2(2m) = N_2(2m, 2) = 4\xi(m); \qquad \xi(2^a m) = \xi(m),$$

the last from the definitions in § 3. Having taken the census and tabulated the primitives for a given r, as in (1), (2), (3) above, we then consider (I)–(VII) of § 9, or so many of them as may be relevant to the particular r, here 3, and substitute successively $s = 1, 2, 3, \cdots$ until the total number of possible squares is exhausted, examining at each step the legitimate forms for n or m in $N_r(n, s+1)$, $N_r(n, s)$, etc., effecting this by an inspection of the census; and last, by referring to the primitives, such as (1)–(3) above, find the proper form for the right hand member of (I), (II), $\cdots$. Thus, putting $r = 3$, $s = 1$ in § 9 (I), the left becomes $3N_3(n, 2)$; and since by the census $N_3(n, 2)$ exists only when $n = 2m$, we must have on the right terms of the form $N_2(2m - \mu^2, 1)$, whose value, by (2) above, is $4\xi(2m - \mu^2)$, the variable $2m - \mu^2$ being odd and of the form $4k + 1$ as required by (2). Proceeding thus with all of the formulas (I)–(VII), we find the following cases, in which, *as always henceforth*, the integer in the numbering (2.1), (3.1), etc., of the formulas indicates from which of the primitive representations it has been derived, and the first decimal the particular one of (I)–(VII) used in the derivation. The rest of the numbering is self-explanatory.

Case I. $\qquad (s+1)N_3(n, s+1) = 6\Sigma N_2(n - \mu^2, s).$

$$(2.1) \qquad N_3(2m) = 12\Sigma\xi(2m - \mu^2);$$

$$(3.1) \qquad m = 8k+3: \quad N_3(m) = 8\Sigma\xi\left(\frac{m - \mu^2}{2}\right).$$

Case II. $(3 - s)N_3(n, s) = 3[N_2(n, s) + 2\Sigma N_2(n - 4a^2, s)]$.

Whence, for $s = 1, 2$:

$$(2.2) \qquad m = 4k + 1: \quad N_3(m) = 6[\xi(m) + 2\Sigma\xi(m - 4a^2)];$$

$$(3.2) \qquad N_3(2m) = 12[\xi(m) + 2\Sigma\xi(m - 2a^2)].$$

The special form $4k + 1$ of m restricts only the formula (2.2) with which it is written, having no relation to subsequent formulas; and so in all similar cases.

Case III. $m = 4k + 1$: $N_3(m, 1) = 6\left[\epsilon(m) + \Sigma N_2\left(\frac{m - \mu^2}{4}\right)\right]$.

$$(1.3) \qquad m = 4k + 1: \quad N_3(m) = 6\left[\epsilon(m) + 4\Sigma\xi\left(\frac{m - \mu^2}{4}\right)\right].$$

Case IV. $N_3(n) = 2\epsilon(n) + N_2(n) + 2\Sigma N_2(n - a^2)$.

$$(1.4) \qquad N_3(n) = 2[\epsilon(n) + 2\xi(n) + 4\Sigma\xi(n - a^2)].$$

Case V. $N_3(4n) = 2[\epsilon(n) + 2\xi(n) + 4\Sigma\xi(n - a^2)]$.

Changing n into $2n$, applying the identity in § 2 in the form

$$\Sigma\xi(2n - a^2) = \Sigma\xi(2n - \mu^2) + \Sigma\xi(2n - 4a^2),$$

and noting that $\xi(2b) = \xi(b)$, we get, as an alternative to the special case of (1.4) in which n is a multiple of 8;

$$(4) \qquad N_3(8n) = 2[\epsilon(2n) + 2\xi(n) + 4\Sigma\xi(2n - \mu^2) + 4\Sigma\xi(n - 2a^2)];$$

and for $n = m$,

$$(5) \quad N_3(4m) = 2\left[\epsilon(m) + 2\xi(m) + 4\Sigma\xi\left(\frac{m - \mu^2}{2}\right) + 4\Sigma\xi(m - 4a^2)\right];$$

both of which illustrate the way in which recurrences for the primitives may be written down on comparing with the equivalent forms deduced from the general (1.4).

IV. Five Squares.*

14. The census for 5 squares is

$N_5(4k) = N_5(4k, 0) + N_5(4k, 4)$; $\quad N_5(8k + 1) = N_5(8k + 1, 1)$;

$N_5(2m) = N_5(2m, 2)$; $\quad N_5(4k + 3) = N_5(4k + 3, 3)$;

$N_5(8k + 5) = N_5(8k + 5, 1) + N_5(8k + 5, 5)$.

Write $\lambda_r(n) = [2(-1)^n + 1]\zeta_r'(n)$, whence

$$\lambda_1(2n) = 3\zeta_1'(n), \; \lambda_1(m) = -\zeta_1(m);$$

* From the formulas (2.1), (6.22), (1.1), (7.2) of cases I, II we get Liouville's results (*J. des Math.*, (2), 4, p. 8); but not conversely.

then the primitive cases for 4 squares as given by Glaisher are:

(1) $N_4(2m, 2) = 24\zeta_1(m)$; (2) $N_4(4m, 4) = 16\zeta_1(m)$;

(3) $N_4(4n, 0) = 8(-1)^n\lambda_1(n) = N_4(n)$; (4) $N_4(2^a m) = 24\zeta_1(m)$;

(5) $N_4(m) = 8\zeta_1(m)$; (6) $m = 4k+1$, $N_4(m) = N_4(m, 1)$;

(7) $m = 4k+3$, $N_4(m) = N_4(m, 3)$.

Case I. $(s+1)N_5(n, s+1) = 10\Sigma N_4(n-\mu^2, s)$.

whence, for $s = 1, 2, 3, 4$:

(6.1) $$N_5(2m) = 40\Sigma\zeta_1(2m-\mu^2);$$

(1.1) $$m = 4k+3: \quad N_5(m) = 80\Sigma\zeta_1\left(\frac{m-\mu^2}{2}\right);$$

(7.1) $$N_5(4n, 4) = 20\Sigma\zeta_1(4n-\mu^2);$$

(2.1) $$m = 8k+5: \quad N_5(m, 5) = 32\Sigma\zeta_1\left(\frac{m-\mu^2}{4}\right).$$

The last was stated by Stieltjes, cf. § 7.

Case II. $(5-s)N_5(n, s) = 5[N_4(n, s) + 2\Sigma N_4(n-4a^2, s)]$.
Putting $s = 1, 2, 3, 4$, we get:

(6.21) $$m = 8k+1: \quad N_5(m) = 10[\zeta_1(m) + 2\Sigma\zeta_1(m-4a^2)];$$

(6.22) $$m = 8k+5: \quad N_5(m, 1) = 10[\zeta_1(m) + 2\Sigma\zeta_1(m-4a^2)];$$

(1.2) $$N_5(2m) = 40[\zeta_1(m) + 2\Sigma\zeta_1(m-2a^2)];$$

(7.2) $$m + 4k + 3: \quad N_5(m) = 20[\zeta_1(m) + 2\Sigma\zeta_1(m-4a^2)];$$

(2.21) $$N_5(4m, 4) = 80[\zeta_1(m) + 2\Sigma\zeta_1(m-4a^2)];$$

(2.22) $$N_5(8n, 4) = 160\Sigma\zeta_1(2n-\mu^2).$$

Case III. $m = 4k+1$: $N_5(m, 1) = 10\left[\epsilon(m) + \Sigma N_4\left(\frac{m-\mu^2}{4}\right)\right]$.

(3.31) $$m = 8k+1: \quad N_5(m) = 10\left[\epsilon(m) + 24\Sigma\zeta_1'\left(\frac{m-\mu^2}{8}\right)\right],$$

(3.32) $$m = 8k+5: \quad N_5(m, 1) = 80\Sigma\zeta_1\left(\frac{m-\mu^2}{2}\right).$$

Cases IV, V. $N_4(4n, 0) = 2\epsilon(n) + N_4(4n, 0) + 2\Sigma N_4(4n-4a^2, 0)$.

(3.4) $$N_5(n) = N_5(4n, 0) = 2\epsilon(n) + 8(-1)^n[\lambda_1(n) + 2\Sigma(-1)^a\lambda_1(n-a^2)].$$

15. From the census, $N_5(4k) = N_5(4k, 0) + N_5(4k, 4)$; hence from

(7.1), (3.4) we get an alternative form of $N_5(4n)$ which may be compared with that given by (3.4). Similarly combining the formulas (2. 1), (3.32), we have

$$(8) \qquad m = 8k + 5: \ N_5(m) = 112\Sigma\zeta_1\left(\frac{m-\mu^2}{4}\right).$$

V. SEVEN SQUARES.

16. The census is

$$\begin{aligned} N_7(4k) &= N_7(4k, 0) + N_7(4k, 4); \\ N_7(2m) &= N_7(2m, 2) + N_7(2m, 6); \\ N_7(8k+7) &= N_7(8k+7, 3) + N_7(8k+7, 7); \\ N_7(4k+1) &= N_7(4k+1, 1) + N_7(4k+1, 5); \\ N_7(8k+3) &= N_7(8k+3, 3); \end{aligned}$$

also

$$\xi'_r(m) = (-1\,|\,m)\xi_r(m),$$

and from Glaisher's lists the primitives for 6 squares are:

(1) $N_6(2m, 2) = 60\xi'_2(m)$; (2) $m = 4k+3: N_6(m, 3) = -20\xi_2(m)$;

(3) $N_6(4n, 4) = 240\xi'_2(n)$; (4) $m = 4k+3: N_6(2m, 6) = -8\xi_2(m)$;

$$(5) \quad N_6(n) = N_6(4n, 0) = 4[4\xi'_2(n) - \xi_2(n)].$$

Case I. $\qquad (s+1)N_7(n, s+1) = 14\Sigma N_6(n-\mu^2, s).$

Putting $s = 2, 3, 4, 6$, we find:

$$(1.1) \qquad m = 4k+3: \ N_7(m, 3) = 280\Sigma\xi'_2\left(\frac{m-\mu^2}{2}\right);$$

$$(1.11) \qquad m = 8k+3: \quad N_7(m) = 280\Sigma\xi_2\left(\frac{m-\mu^2}{.2}\right);$$

$$(1.12) \qquad m + 8k + 7: \ N_7(m, 3) = -280\Sigma\xi_2\left(\frac{m-\mu^2}{2}\right);$$

$$(2.1) \qquad N_7(4n, 4) = -70\Sigma\xi_2(4n-\mu^2);$$

$$(3.1) \qquad m = 4k+1: \ N_7(m, 5) = 672\Sigma\xi'_2\left(\frac{m-\mu^2}{4}\right);$$

$$(4.1) \qquad m = 8k+7: \ N_7(m, 7) = -16\Sigma\xi_2\left(\frac{m-\mu^2}{2}\right).$$

From (1.12), (4.1) we see that for $m \equiv 7 \bmod 8$, $2N_7(m, 3) = 35N_7(m, 7)$. Many such relations may be read off from the lists for 3, $\cdots$, 13 squares.

Case II. $(7 - s)N_7(n, s) = 7[N_6(n, s) + 2\Sigma N_6(n - 4a^2, s)]$.
Putting $s = 2, 3, 4, 6$:

(1.2) $N_7(2m, 2) = 84[\xi_2'(m) + 2\Sigma\xi_2'(m - 2a^2)]$;

(2.21) $m = 8k + 3$: $N_7(m) = -35[\xi_2(m) + 2\Sigma\xi_2(m - 4a^2)]$;

(2.22) $m = 8k + 7$: $N_7(m, 3) = -35[\xi_2(m) + 2\Sigma\xi_2(m - 4a^2)]$;

(3.2) $N_7(4n, 4) = 560[\xi_2'(n) + 2\Sigma\xi_2'(n - a^2)]$;

(4.2) $m = 4k + 3$: $N_7(2m, 6) = -56[\xi_2(m) + 2\Sigma\xi_2(m - 8a^2)]$.

Case III. $m = 4k + 1$: $N_7(m, 1) = 14\left[\epsilon(m) + \Sigma N_6\left(\frac{m - \mu^2}{2}\right)\right]$.

(5.3) $m=4k+1$: $N_7(m, 1)=14\left[\epsilon(m)+4\Sigma\left\{4\xi_2'\left(\frac{m-\mu^2}{4}\right)-\xi_2\left(\frac{m-\mu^2}{4}\right)\right\}\right]$.

Cases IV, V. $N_7(4n, 0) = N_7(n) =$

(5.4) $2\epsilon(n) + 4[4\xi_2'(n) - \xi_2(n)] + 8\Sigma[4\xi_2'(n - a^2) - \xi_2(n - a^2)]$.

The equivalent of (5.4) was stated by Stieltjes, *C.R.*, 31 Dec., 1884.

Cases VI, VII. For the first time these enter. From Glaisher's theorems for 6 squares (loc. cit., p. 10), we find on eliminating non-primitives,

(7) $m = 4k + 1$: $N_6(m, 1) + N_6(m, 5) = 12\xi_2(m)$.

Hence, taking § 11 (VI) in the form

$$(1 + s_1)N_7(n, 1 + s_1) + (1 + s_2)N_7(n, 1 + s_2)$$
$$= 14\Sigma[N_6(n - \mu^2, s_1) + N_6(n - \mu^2, s_2)],$$

and choosing $(s_1, s_2) = (1, 5)$, we get, on referring to the census,

(7.6) $m = 4k + 1$: $N_7(2m, 2) + 3N_7(2m, 6) = 84\Sigma\xi_2(2m - \mu^2)$;

and from (VII) in the same way,

(7.7) $m = 4k + 1$: $3N_7(m, 1) + N_7(m, 5) = 42[\xi_2(m) + 2\Sigma\xi_2(m - 4a^2)]$.

Combining (7.7), (3.1) or (7.7), (5, 3) we find alternative forms for $N_7(m, 1)$ or $N_7(m, 5)$ ($m \equiv 1 \bmod 4$) respectively; and from (7.6), (1.2):

(8) $m = 4k + 1$: $N_7(2m, 6) = 28[\Sigma\xi_2(2m - \mu^2) - \xi_2(m) - 2\Sigma\xi_2'(m - 2a^2)]$,

which is at once expressible in terms of ξ_2' alone on using the identity of § 2.

17. From the lists in § 16 we write down the following additional complete formulas by reference to the census. Some, such as $N_7(4n)$, which are not much better adapted to computation than those given directly by (5.4), have been omitted.

$$(9) \quad m = 4k+1: \quad N_7(m) = 14\left[\epsilon(m) + 64\Sigma\xi_2'\left(\frac{m-\mu^2}{4}\right) - 4\Sigma\xi_2\left(\frac{m-\mu^2}{4}\right)\right];$$

$$(10) \quad m = 4k+1: \; N_7(2m) = 28[\xi_2(m) + \Sigma\xi_2(2m-\mu^2) + 4\Sigma\xi_2'(m-2a^2)];$$

$$(11) \quad m = 4k+3: \; N_7(2m) = -28[5\xi_2(m) - 6\Sigma\xi_2(m-2\mu^2) + 10\Sigma\xi_2(m-8a^2)];$$

$$(12) \quad m = 8k+7: \quad N_7(m) = -296\Sigma\xi_2\left(\frac{m-\mu^2}{2}\right).$$

VI. NINE SQUARES.

18. The census is

$$
\begin{aligned}
N_9(4n) &= N_9(4n, 0) + N_9(4n, 4) + N_9(4n, 8),\\
m = 8k+1: \; N_9(m) &= N_9(m, 1) + N_9(m, 5) + N_9(m, 9),\\
N_9(2m) &= N_9(2m, 2) + N_9(2m, 6),\\
m = 8k+5: \; N_9(m) &= N_9(m, 1) + N_9(m, 5),\\
m = 4k+3: \; N_9(m) &= N_9(m, 3) + N_9(m, 7).
\end{aligned}
$$

To state the primitives for 8 squares it is convenient to introduce $\rho_3(n)$, $\alpha_3(n)$, where $\rho_r(n) = \zeta_r'(n) - \zeta_r''(n)$, and $\alpha_r(n) = n^r\zeta_{-r}'(n) =$ the sum of the rth powers of all those divisors of n whose conjugates are odd. From these definitions we have at once the useful identities:

$$
\begin{aligned}
&\alpha_3(m) = \zeta_3(m), \qquad \alpha_3(2m) = 8\zeta_3(m), \qquad \alpha_3(2n) = 8\alpha_3(n),\\
&\rho_3(m) = \zeta_3(m), \qquad \rho_3(2m) = -7\zeta_3(m), \qquad \rho_3(2n) = 8\rho_3(n) - 15\zeta_3'(n),\\
&\qquad 7\rho_3(n) + 8\alpha_3(n) = 15\zeta_3'(n).
\end{aligned}
$$

The primitives for 8 squares, from Glaisher, are:

(1) $N_8(4n, 4) = 1120\alpha_3(n)$; (2) $N_8(8n, 8) = 256\alpha_3(n)$;

(3) $N_8(n) = -16(-1)^n\rho_3(n)$; (4) $N_8(4n, 0) = N_8(n)$.

Case I. $(s+1)N_9(n, s+1) = 18\Sigma N_8(n-\mu^2, s)$.

Whence, for $s = 4, 8$:

$$(1.1) \qquad m = 4k+1: \; N_9(m, 5) = 4032\Sigma\alpha_3\left(\frac{m-\mu^2}{4}\right);$$

$$(1.11) \qquad m = 8k+1: \; N_9(m, 5) = 32256\Sigma\alpha_3\left(\frac{m-\mu^2}{8}\right);$$

$$(1.12) \qquad m = 8k+5: \; N_9(m, 5) = 4032\Sigma\zeta_3\left(\frac{m-\mu^2}{4}\right);$$

$$(2.1) \qquad m = 8k+1: \; N_9(m, 9) = 512\Sigma\alpha_3\left(\frac{m-\mu^2}{8}\right).$$

Case II. $(9 - s)N_9(n, s) = 9[N_8(n, s) + 2\Sigma N_8(n - 4a^2, s)]$.

Putting $s = 4, 8$, and separating the cases according as $m \equiv 0$ or $4 \bmod 8$, we find, after some short reductions:

(1.2) $N_9(4n, 4) = 2016[\alpha_3(n) + 2\Sigma\alpha_3(n - a^2)]$;

(1.21) $N_9(4m, 4) = 2016\left[\zeta_3(m) + 2\Sigma\zeta_3(m - 4a^2) + 16\Sigma\alpha_3\left(\frac{m - \mu^2}{2}\right)\right]$;

(1.22) $N_9(8n, 4) = 4032[4\alpha_3(n) + 8\Sigma\alpha_3(n - 2a^2) + \Sigma\zeta_3(2n - \mu^2)]$;

(2.21) $N_9(4m, 8) = 4608\Sigma\alpha_3\left(\frac{m - \mu^2}{2}\right)$;

(2.22) $N_9(8n, 8) = 2304[\alpha_3(n) + 2\Sigma\alpha_3(n - 2a^2)]$.

Case III. $m = 4k + 1$: $N_9(m, 1) = 18\left[\epsilon(m) + \Sigma N_8\left(\frac{m - \mu^2}{4}\right)\right]$.

(3.31) $m = 8k + 1$: $N_9(m, 1)$

$$= 18\left[\epsilon(m) + 240\Sigma\zeta_3'\left(\frac{m - \mu^2}{8}\right) - 128\Sigma\rho_3\left(\frac{m - \mu^2}{8}\right)\right];$$

(3.32) $m = 8k + 5$: $N_9(m, 1) = 288\Sigma\zeta_3\left(\frac{m - \mu^2}{4}\right)$.

Cases IV, V. $N_9(n) = N_9(4n, 0) =$

(3.4) $2\epsilon(n) - 16(-1)^n[\rho_3(n) + 2\Sigma(-1)^a\rho_3(n - a^2)]$.

Cases VI, VII. From Glaisher's results for 8 squares we have:

(5) $m = 4k + 1$: $N_8(m, 1) + N_8(m, 5) = 16\zeta_3(m)$;

(6) $N_8(2m, 2) + N_8(2m, 6) = 112\zeta_3(m)$;

(7) $m = 4k + 3$: $N_8(m, 3) + N_8(m, 7) = 16\zeta_3(m)$.

Hence, putting $(s_1, s_2) = (1, 5), (2, 6), (3, 7)$ in (VI), (VII) of § 11, and comparing as usual with the census and primitives:

(5.6) $N_9(2m, 2) + 3N_9(2m, 6) = 144\Sigma\zeta_3(2m - \mu^2)$;

(6.6) $m = 4k + 3$: $3N_9(m, 3) + 7N_9(m, 7) = 2016\Sigma\zeta_3\left(\frac{m - \mu^2}{2}\right)$;

(7.6) $N_9(4n, 4) + 2N_9(4n, 8) = 72\Sigma\zeta_3(4n - \mu^2)$;

(5.7) $m = 4k + 1$: $2N_9(m, 1) + N_9(m, 5) = 36[\zeta_3(m)$

$+ 2\Sigma\zeta_3(m - 4a^2)]$;

$$(6.7)\qquad 7N_9(2m,\,2)+3N_9(2m,\,6)=1008[\zeta_3(m)+2\Sigma\zeta_3(m-2a^2)];$$

$$(7.7)\qquad m=4k+3:\ 3N_9\ (m,\,3)+\ N_9\ (m,\,7)=\ 72[\zeta_3(m)+2\Sigma\zeta_3(m-4a^2)].$$

Solving these we get:

$$(9)\qquad N_9(2m,\,2)=24[7\zeta_3(m)+14\Sigma\zeta_3(m-2a^2)-\Sigma\zeta_3(2m-\mu^2)];$$

$$(10)\qquad N_9(2m,\,6)=56[-\zeta_3(m)-2\Sigma\zeta_3(m-2a^2)+\Sigma\zeta_3(2m-\mu^2)];$$

(11) $m=4k+3$:

$$N_9\ (m,\,3)=28\left[\zeta_3(m)+2\Sigma\zeta_3(m-4a^2)-4\Sigma\zeta_3\left(\frac{m-\mu^2}{2}\right)\right];$$

(12) $m=4k+3$:

$$N_9\ (m,\,7)=12\left[-\zeta_3(m)-2\Sigma\zeta_3(m-4a^2)+28\Sigma\zeta_3\left(\frac{m-\mu^2}{2}\right)\right].$$

Similarly from (1.1), (5.7):

(13) $m=4k+1$:

$$N_9(m,\,1)=18\left[\zeta_3(m)+2\Sigma\zeta_3(m-4a^2)-112\Sigma\alpha_3\left(\frac{m-\mu^2}{4}\right)\right].$$

19. Combining certain of the formulas in § 18 according to the census we find the additional complete cases in the following list. Thus, the value of $N_9(2m)=N_9(2m,\,2)+N_9(2m,\,6)$ follows from (9), (10). Some that may be found in this way have been omitted.

(14) $m=8k+1$:

$$N_9\ (m)=18\zeta_3(m)+36\Sigma\zeta_3(m-4a^2)+16640\Sigma\alpha_3\left(\frac{m-\mu^2}{8}\right);$$

(15) $m=8k+5$:

$$N_9\ (m)=4320\Sigma\zeta_3\left(\frac{m-\mu^2}{4}\right);$$

(16) $m=4k+3$:

$$N_9\ (m)=16\left[\zeta_3(m)+2\Sigma\zeta_3(m-4a^2)+14\Sigma\zeta_3\left(\frac{m-\mu^2}{2}\right)\right];$$

$$(17)\qquad N_9(2m)=16[7\zeta_3(m)+14\Sigma\zeta_3(m-2a^2)+2\Sigma\zeta_3(2m-\mu^2)].$$

VII. ELEVEN SQUARES.*

20. The census is in § 5; and the primitives for ten squares were found by Glaisher to be:

* The references in Bachmann's Zahlentheorie, Bd. IV, and the German Encyclopaedia to Liouville's consideration of this case are misprints; Liouville published nothing concerning 11 squares.

(1) $5N_{10}(n) = 4[\xi_4(n) + 16\xi_4'(n)]$; (2) $N_{10}(2m, 2) = -\ 36\xi_4(m)$;

(3) $N_{10}(4n, 4) = 2688\xi_4'(n)$; (4) $N_{10}(2m, 6) = -\ 168\xi_4(m)$;

(5) $N_{10}(4n, 8) = 576\xi_4'(n)$; (6) $N_{10}(4n, 0) = N_{10}(n)$,

in all of which $n = 2^b m$, $m = 4k + 3$, $b \geqq 0$. For the same value of n, $\xi_4'(n) = -\ 2^{4b}\xi_4(m)$, $\xi_4(n) = \xi_4(m)$; and the numbers $n - a^2$ ($a = 1, 2, 3, \cdots$) are all of the form $2^c(4k + 3)$, $c \geqq 0$, when and only when $b = 0, 2$, and $m = 8k + 7$. Hence we find only the following simple expressions for 11 squares.

Case I. $(s + 1)N_{11}(n, s + 1) = 22\Sigma N_{10}(n - \mu^2, s)$.

Whence, for $s = 2, 6$:

$$(2.1)\qquad m = 8k + 7:\ N_{11}(m, 3) = -\ 264\Sigma\xi_4\left(\frac{m - \mu^2}{2}\right);$$

$$(4.1)\qquad m = 8k + 7:\ N_{11}(m, 7) = -\ 528\Sigma\xi_4\left(\frac{m - \mu^2}{2}\right).$$

Case II. $(11 - s)N_{11}(n, s) = 11[N_{10}(n, s) + 2\Sigma N_{10}(n - 4a^2, s)]$.

Comparing this with (3), (5), we see that when and only when n is of either form $8k + 7$, $4(8k + 7)$,

$$(3.2)\qquad N_{11}(4n, 4) = 4224[\xi_4'(n) + 2\Sigma\xi_4'(n - a^2)];$$

$$(5.2)\qquad N_{11}(4n, 8) = 2112[\xi_4'(n) + 2\Sigma\xi_4'(n - a^2)].$$

Case IV. $N_{11}(n) = 2\epsilon(n) + N_{10}(n) + 2\Sigma N_{10}(n - a^2)$.

Hence for n as in case II, and $m = 8k + 7$:

$$(1.4)\qquad 5N_{11}(n) = 4[\xi_4(n) + 16\xi_4'(n) + 2\Sigma\{\xi_4(n - a^2) + 16\xi_4'(n - a^2)\}];$$

$$(1.41)\qquad N_{11}(m) = -\ 12\left[\xi_4(m) + 2\Sigma\xi_4(m - 4a^2) + 32\Sigma\xi_4\left(\frac{m - \mu^2}{2}\right)\right].$$

21. Combining (2.1), (4.1) by the census,

$$(7)\qquad m = 8k + 7:\ N_{11}(m) = -\ 792\Sigma\xi_4\left(\frac{m - \mu^2}{2}\right).$$

The census gives also $N_{11}(4m) = N_{11}(4m, 0) + N_{11}(4m, 4) + N_{11}(4m, 8)$, and (1.4) the value of $N_{11}(4m, 0)$; hence from these and (3.2), (5.2) may be derived recurrences for the ξ_4, ξ_4'; and similarly from (1.4) and (7).

22. By eliminating the non-primitives from the formulas for $N_{10}(n, r)$ as given by Glaisher, it is easy to deduce several relations of the forms (VI), (VII) of § 11, and therefore to find simple linear functions of N's. But, as in preceding sections, it is verified without difficulty that the com-

plete system of such relations when combined with the formulas of §§ 20, 21 is insufficient for the determination of any N_{11} not already listed. This verification, presenting nothing new, is omitted. The like may be shown to hold for 13, 15, 17, 19, 21, 23, 25 squares; and when we pass to 13 squares (the next case considered) there is the concurrent disappearance of primitives for the associated even number of squares. Whether both or either of these failures occurs always for m squares when > 25, has been shown in another paper to depend upon the divisors common to certain binomial coefficients and the numerators of the numerical coefficients in the power series for the Jacobian elliptic functions, and need only be mentioned here. We remark, however, that one aspect of the failure is permanent after 13 squares; viz., the number of linear relations between the several $N_m(n, r)$ is always less than the number of unknown functions.

VIII. THIRTEEN SQUARES.

23. It is unnecessary to take the census. $\alpha_r(n)$ is defined in § 18; and we now introduce $\beta_r(n)$, $= \zeta'_r(n) - \zeta''_r(n) + 2\alpha_r(n)$. This function is susceptible of many transformations; in particular it may be expressed in terms of 2^s and $\zeta_r(m)$, where $n = 2^s m$, but the form stated is sufficient for our purpose. The only primitives for 12 squares are

$$(1)\ N_{12}(2n) = 8\beta_5(2n); \qquad (2)\ N_{12}(8n, 4) = N_{12}(8n, 8) = 126720\alpha_5(n).$$

An equivalent form of (1) was stated with insufficient proof by Liouville,* and first proved by Glaisher, from whose paper both are transcribed. We shall consider the primitive $N_{12}(8n, 0)$ as included under (1). It is readily seen that Cases I, III alone are applicable.

Case I. $\qquad (s + 1)N_{13}(n, s + 1) = 26\Sigma N_{12}(n - \mu^2, s).$

Whence, for $s = 4, 8$:

$$(2.11) \qquad m = 8k + 1:\ N_{13}(m, 5) = 658944\Sigma\alpha_5\left(\frac{m - \mu^2}{8}\right);$$

$$(2.12) \qquad m = 8k + 1:\ N_{13}(m, 9) = 366080\Sigma\alpha_5\left(\frac{m - \mu^2}{8}\right).$$

Hence $5N_{13}(m, 5) = 9N_{13}(m, 9)$ when $m \equiv 1 \bmod 8$.

* *Journal des Math.*, (2) 6 (1861), p. 206. The proof is insufficient because it is made to depend upon a theorem which Liouville did not prove, and whose proof is more difficult than that of the special result for 12 squares; cf. Bachmann, Zahlentheorie, Bd. IV, p. 668. The same applies also to the primitives for 10 squares, first stated by Eisenstein and Liouville, but first proved by Glaisher.

Case III. $m = 4k + 1$: $N_{13}(m, 1) = 26\left[\epsilon(m) + \Sigma N_{12}\left(\frac{m - \mu^2}{4}\right)\right].$

(1.3) $m = 8k + 1$: $N_{13}(m, 1) = 26\left[\epsilon(m) + 8\Sigma\beta_5\left(\frac{m - \mu^2}{4}\right)\right].$

24. The census gives

$$m = 8k + 1: \ N_{13}(m) = N_{13}(m, 1) + N_{13}(m, 5) + N_{13}(m, 9).$$

Hence, from § 23, we have

(3) $m = 8k + 1$:

$$N_{13}(m) = 26\left[\epsilon(m) + 8\Sigma\beta_5\left(\frac{m - \mu^2}{4}\right) + 39424\Sigma\alpha_5\left(\frac{m - \mu^2}{8}\right)\right].$$

IX. RECURRENCES.

25. In the numbering (3.1), etc., of the following formulas the integer $= r$, and the first decimal $= s$, and the resulting recurrence is written down from the general formula at the head of its set by substituting the values of (r, s) thus defined. *E.g.*, we get the first (3.1) on putting $r = 3$ $s = 1$ in (I′), and referring to § 14 for the value of

$$(s + 1)N_{r+1}(m + 1, s + 1), = 2N_4(m + 1, 2),$$

noting that $m + 1$ is an even integer, say $2n$; hence $m = 2n - 1$, and therefore being any odd positive integer, is denoted by m in (3.1). Similarly all the recurrences under the several cases (I′)–(IV′) are written down by inspection on glancing first at the primitives for $N_{r+1}(m + 1, s + 1)$, $N_{r+1}(n, s)$, $N_{r+1}(4n + 1, 1)$, $N_{r+1}(n)$ respectively, and giving r the values 3, 5, 7, 9, 11, in succession. We recall that t represents always a triangular number. Cases I′–IV′ are from § 12.

Case I′. $2(r + 1)N_r(n, s) = (s + 1)N_{r+1}(n + 1, s + 1) - 2(r + 1)\Sigma N_r(n - 8t, s).$

(3.1) $m = 4k + 1$: $N_3(m) = 6\zeta_1\left(\frac{m + 1}{2}\right) - \Sigma N_3(m - 8t);$

(3.2) $N_3(2m) = 3\zeta_1(2m + 1) - \Sigma N_3(2m - 8t);$

(3.3) $m = 8k + 3$: $N_3(m) = 8\zeta_1\left(\frac{m + 1}{4}\right) - \Sigma N_3(m - 8t);$

(5.11) $m = 8k + 1$: $N_5(m) = 10\zeta_2\left(\frac{m + 1}{2}\right) - \Sigma N_5(m - 8t);$

(5.12) $m = 8k + 5$: $N_5(m, 1) = -10\xi_2\left(\frac{m + 1}{2}\right) - \Sigma N_5(m - 8t, 1);$

(5.2) $$N_5(2m) = -5\xi_2(2m+1) - \Sigma N_5(2m-8t);$$

(5.3) $m = 4k+3$: $$N_5(m) = 80\xi_2'\left(\frac{m+1}{4}\right) - \Sigma N_5(m-8t);$$

(5.5) $m = 8k+5$: $$N_5(m, 5) = -4\xi_2\left(\frac{m+1}{2}\right) - \Sigma N_5(m-8t, 5);$$

(5.51) $m = 8k+5$: $$N_5(m) = -14\xi_2\left(\frac{m+1}{2}\right) - \Sigma N_5(m-8t);$$

(7.31) $m = 8k+3$: $$N_7(m) = 280\alpha_3\left(\frac{m+1}{4}\right) - \Sigma N_7(m-8t);$$

(7.32) $m = 8k+7$: $$N_7(m, 3) = 2240\alpha_3\left(\frac{m+1}{8}\right) - \Sigma N_7(m-8t, 3);$$

(7.7) $m = 8k+7$: $$N_7(m, 7) = 128\alpha_3\left(\frac{m+1}{8}\right) - \Sigma N_7(m-8t, 7);$$

(7.71) $m = 8k+7$: $$N_7(m) = 2368\alpha_3\left(\frac{m+1}{8}\right) - \Sigma N_7(m-8t);$$

(9.1) $m = 8k+5$: $$5N_9(m, 1) = -18\xi_4\left(\frac{m+1}{2}\right) - 5\Sigma N_9(m-8t, 1);$$

(9.31) $m = 16k+11$: $$5N_9(m, 3) = 2688\xi_4'\left(\frac{m+1}{4}\right) - 5\Sigma N_9(m-8t, 3);$$

(9.5) $m = 8k+5$: $$5N_9(m, 5) = -252\xi_4\left(\frac{m+1}{2}\right) - 5\Sigma N_9(m-8t, 5);$$

(9.51) $m = 8k+5$: $$N_9(m) = -54\xi_4\left(\frac{m+1}{2}\right) - \Sigma N_9(m-8t);$$

(9.7) $m = 16k+11$: $$5N_9(m, 7) = 1152\xi_4'\left(\frac{m+1}{4}\right) - 5\Sigma N_9(m-8t, 7);$$

(9.71) $m = 16k+11$: $$N_9(m) = 768\xi_4'\left(\frac{m+1}{4}\right) - \Sigma N_9(m-8t);$$

(11.3) $m = 8k+7$: $$N_{11}(m, 3) = 21120\alpha_5\left(\frac{m+1}{8}\right) - \Sigma N_{11}(m-8t, 3);$$

(11.7) $m = 8k+7$: $$N_{11}(m, 7) = 44240\alpha_5\left(\frac{m+1}{8}\right) - \Sigma N_{11}(m-8t, 7);$$

(11.71) $m = 8k+7$: $$N_{11}(m) = 63360\alpha_5\left(\frac{m+1}{8}\right) - \Sigma N_{11}(m-8t).$$

Case II'. $(r+1)N_r(n,s) = (r+1-s)N_{r+1}(n,s)$
$- 2(r+1)\Sigma N_r(n-4a^2,s).$

(3.1) $m = 4k+1$: $N_3(m) = 6\zeta_1(m) - 2\Sigma N_3(m-4a^2)$;

(3.2) $N_3(2m) = 12\zeta_1(m) - 2\Sigma N_3(2m-4a^2)$;

(3.3) $m = 8k+3$: $N_3(m) = 2\zeta_1(m) - 2\Sigma N_3(m-4a^2)$;

(5.2) $N_5(2m) = 40\xi_2'(m) - 2\Sigma N_5(2m-4a^2)$;

(5.3) $m = 4k+3$: $N_5(m) = -10\xi_2(m) - 2\Sigma N_5(m-4a^2)$;

(5.4) $N_5(4n,4) = 80\xi_2'(n) - 2\Sigma N_5(4n-4a^2,4)$;

(7.4) $N_7(4n,4) = 560\alpha_3(n) - 2\Sigma N_7(4n-4a^2,4)$;

(9.2) $m = 4k+3$: $5N_9(2m,2) = -144\xi_4(m) - 10\Sigma N_9(2m-4a^2,2)$;

(9.4) $n = 2^b(4k+3)$, $b \geqq 0$:
$5N_9(4n,4) = 8064\xi_4'(n) - 10\Sigma N_9(4n-4a^2,4)$;

(9.6) $m = 4k+3$: $5N_9(2m,6) = -336\xi_4(m) - 10\Sigma N_9(2m-4a^2,6)$;

(9.61) $m = 4k+3$: $N_9(2m) = -96\xi_4(m) - 10\Sigma N_9(2m-4a^2)$;

(9.8) $n = 2^b(4k+3)$, $b \geqq 0$:
$5N_9(4n,8) = 576\xi_4'(n) - 10\Sigma N_9(4n-4a^2,8)$;

(11.4) $N_{11}(8n,4) = 84480\alpha_5(n) - 2\Sigma N_{11}(8n-4a^2,4)$;

(11.8) $N_{11}(8n,8) = 42240\alpha_5(n) - 2\Sigma N_{11}(8n-4a^2,8)$.

Case III'. $2(r+1)N_r(n) = N_{r+1}(4n+1,1)$
$- 2(r+1)[\epsilon(4n+1) + \Sigma N_r(n-2t)].$

(3.1) $N_3(n) = \zeta_1(4n+1) - \epsilon(4n+1) - \Sigma N_3(n-2t)$.

Case IV'. $N_r(n) = N_{r+1}(n) - 2\epsilon(n) - 2\Sigma N_r(n-a^2)$.

(3) $N_3(n) = 8(-1)^n\lambda_1(n) - 2\epsilon(n) - 2\Sigma N_3(n-a^2)$;

(5) $N_5(n) = 4[\xi_2'(n) - \xi_2(n)] - 2\epsilon(n) - 2\Sigma N_5(n-a^2)$;

(7) $N_7(n) = -16(-1)^n\rho_3(n) - 2\epsilon(n) - 2\Sigma N_7(n-a^2)$;

(9) $n = 2^b(4k+3)$, $b \geqq 0$:
$5N_9(n) = 4[\xi_4(n) + 16\xi_4'(n)] - 10\epsilon(n) - 10\Sigma N_9(n-a^2)$;

(11) $N_{11}(2n) = 8\beta_5(n) - 2\epsilon(2n) - 2\Sigma N_{11}(2n-a^2)$.

Since $n - a^2$ is not always of the form $2^b(4k+3)$, (9) does not enable us

to calculate the values of $N_9(n)$ by recurrence alone; and the same holds for (11), half the values of $2n - a^2$ being odd. The like applies to certain of the formulas under I′–III′; but in conjunction with the results of sections III–VII, even the incomplete recurrences are a material aid to computation.

26. Comparing the finite sums given by $N_r(n)$ under cases IV in sections III–VII with those determined under cases I–III, we may derive useful recurrences for certain of the primitives. These, which are of the same general type as the two examples given by Liouville (cf. § 14, footnote), we omit.

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